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Note

A note on 3-connected cubic planar graphs

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ABSTRACT

The length of a longest cycle in a graph G is called the *circumference* of G and is denoted by $c(G)$. Let $c(n) = \min\{c(G) : G$ is a 3-connected cubic planar graph of order $n\}$. Tait conjectured in 1884 that $c(n) = n$, and Tutte disproved this in 1946 by showing that $c(n) \le n - 1$ for $n = 46$. We prove that the inequality $c(n) \le n - \sqrt{n + \frac{49}{4}} + \frac{5}{2}$ holds for infinitely many integers n. The exact value of $c(n)$ is unknown.

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1. Introduction

The length of a longest cycle in a graph G is called the *circumference* of G and is denoted by $c(G)$. There are many results on the lower bound of $c(G)$. For example, Chen and Yu [2] proved that $c(G) \ge \alpha|G|^{\log_3 2}$ if G is a 3-connected planar graph. Also, in 1956 Tutte [7] proved that every 4-connected planar graph is hamiltonian. On the basis of the Bridge Lemma in [7], we have that $c(G) \ge \frac{3n}{4}$ for every cubic cyclically 4-edge-connected graph. However, there is not much being said about the upper bound of $c(G)$.

Let P be a graph property and $G(P)$ be the set of all graphs with property P. We define the quantity $c(n, P)$ as

$$c(n, P) = \min\{c(G) : |G| = n, G \in G(P)\}.$$

In this paper we focus on the property P that G is a 3-connected cubic planar graph, and we use $c(n)$ instead of $c(n, P)$ for short.

In 1884, Tait made the following conjecture in [5]: If G is a 3-connected cubic planar graph, then G is hamiltonian. In other words, Tait conjectured that $c(n) = n$. Tutte disproved this conjecture by constructing a non-hamiltonian 3-connected cubic planar graph in [6]. This proves that $c(n) \le n - 1$ for some n. See [8] for more detail on how the counterexample was found.

In this paper we improve this upper bound and obtain the following result: there exist 3-connected cubic planar graphs of order n with longest cycle of length at most $n - \sqrt{n + \frac{49}{4}} + \frac{5}{2}$ for infinitely many integers n. In other words, we shall prove that

$$c(n) \le n - \sqrt{n + \frac{49}{4}} + \frac{5}{2}$$

for infinitely many integers n. It is still open to find the exact value for $c(n)$.

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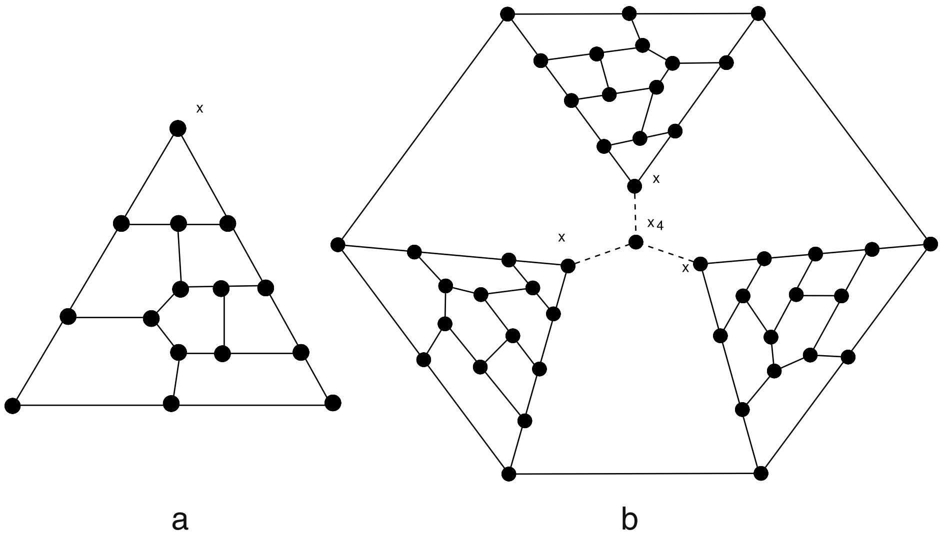

Fig. 1. (a) A Tutte fragment H with critical vertex x. (b) The graph $K_4[\{x_1, x_2, x_3\}, H]$.

Remark. If we drop the requirement that G must be cubic, then the upper bound can be lowered to $cn^{\log_3 2}$. A family of such examples is presented in [4].

All graphs here are simple and planar and we follow the terminology in [1].

An edge e of a graph G is called a *critical edge* if G is hamiltonian but $G - e$ is not.

Let G' be a cubic graph with a vertex v having its three neighbors a, b and c, and $H = G' - v$. Let G be another cubic graph with a vertex u having its three neighbors x, y and z. We construct a new graph $G[u, H]$ as follows: $G[u, H]$ has vertex set $V(G) \cup V(H) - \{u\}$ and edge set $(E(G) - \{ux, uy, uz\}) \cup E(H) \cup \{ax, by, cz\}$. Intuitively, $G[u, H]$ is obtained from G by replacing the vertex u with the graph H. Since ax, by and cz are the only edges that connect H to outside of it, we call them the *outward* edges of H, and we also call the vertices a, b and c the *boundary* vertices of H. The edge ax is the outward edge incident with a. Similarly by is the outward edge incident with b and cz is the outward edge incident with c.

From the construction we see that if G and G' are 3-connected and planar, then $G[u, H]$ is also 3-connected and planar. If $G[u, H]$ has a Hamilton cycle C, then C must contain two segments: one consists of all vertices of H and the other consists of all remaining ones, and these two segments are connected by two outward edges of H. In other words, we may assume that $C = x_1x_2 \dots x_hx_{h+1} \dots x_n$ such that $x_i \in V(H)$ for $1 \leq i \leq h = |V(H)|$ and $x_j \in V(G[u, H]) - V(H)$ for $h + 1 \leq j \leq n$. If we can find a Hamilton cycle C in $G[u, H]$, then we can also obtain a Hamilton cycle of G by contracting the segment in H back into a single vertex. For example, from the above mentioned $C = x_1x_2 \dots x_hx_{h+1} \dots x_n$ we obtain the cycle $C' = ux_{h+1} \dots x_n$ of G by contracting the segment $x_1 \dots x_h$ back into a single vertex u.

Once we know how to replace a single vertex of G with H, we can continue the process and construct a graph by replacing a subset of vertices of G with distinct copies of H. Let $X = \{x_1, x_2, \dots, x_k\} \subseteq V(G)$; we can construct a sequence of new graphs G_i as follows. Let $G_0 = G$, and let $G_i = G_{i-1}[x_i, H]$ for $1 \leq i \leq k$. We say that the final graph G_k is obtained from G by replacing X with $|X|$ distinct copies of H, and denote it by $G[X, H]$. We may use the simpler notation $G[H]$ if we know the set X from the context.

If G' has a critical edge va, then H is called a *Tutte fragment*, and vertex a of H is called a *critical vertex of H*. The corresponding outward edge e incident with a in $G[X, H]$ is called a *pseudo-critical* edge. Any Hamilton cycle of $G[X, H]$ must contain the edge e; thus e is critical if $G[X, H]$ is hamiltonian. Since we are not sure whether $G[X, H]$ is hamiltonian, we call it a pseudo-critical edge instead. A Tutte fragment can be used to construct non-hamiltonian 3-connected cubic planar graphs.

Let H be a Tutte fragment and G_0 a cubic graph with $X \subseteq V(G_0)$. If $G = G_0[X, H]$ has one pseudo-critical edge that cannot be in a longest cycle, then G cannot be hamiltonian. For example, let K_4 be the complete graph of order 4 on vertex set $\{x_1, x_2, x_3, x_4\}$, and $G = K_4[\{x_1, x_2, x_3\}, H]$ such that all critical vertices connect to x_4; then G is non-hamiltonian since it is impossible to have a longest cycle containing all three incident pseudo-critical edges. See Fig. 1 for an illustration. The Tutte fragment H in Fig. 1 is the one used by Tutte in [6]. It is the one obtained by taking $k = 2$ in the construction in the next section.

2. Constructing Tutte fragments

Let C_{2k+1} be a cycle of order $2k + 1$, and let $G_0 = C_{2k+1} \times K_2$ be the prism graph of order $4k + 2$. In other words, G_0 consists of two vertex disjoint cycles of length $2k + 1$, say $C = x_1x_2 \dots x_{2k+1}$ and $C' = y_1y_2 \dots y_{2k+1}$, together with edges

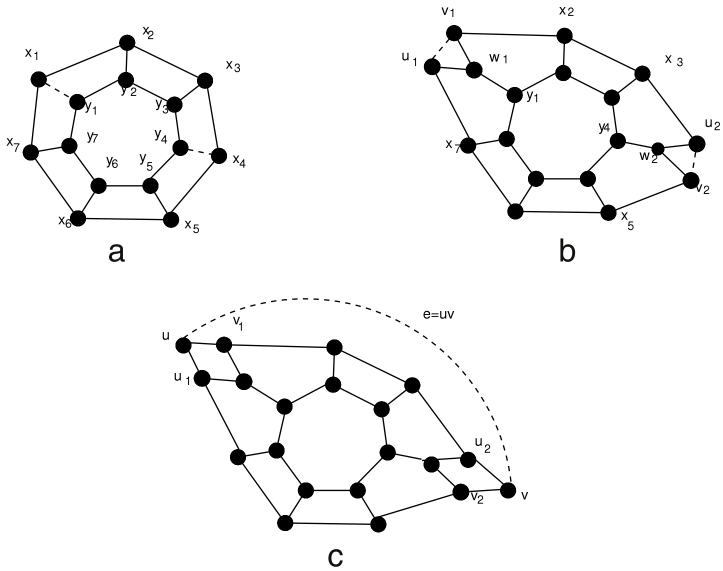

Fig. 2. (a) No Hamilton cycle contains x_1y_1, x_4y_4 simultaneously. (b) No Hamilton cycle contains u_1v_1, u_2v_2 simultaneously. (c) Every Hamilton cycle must contain edge uv.

$x_1y_1, x_2y_2, \ldots, x_{2k+1}y_{2k+1}$. Let $M = \{x_1y_1, x_2y_2, \ldots, x_{2k+1}y_{2k+1}\}$. Then M is a perfect matching of G_0. The edges x_iy_i and $x_{i+1}y_{i+1}$ are called *adjacent edges with respect to M* for $1 \le i \le 2k+1$, with the understanding that $x_{2k+2}y_{2k+2} = x_1y_1$, so x_1y_1 and $x_{2k+1}y_{2k+1}$ are adjacent with respect to M.

We list the following facts as a lemma without proof. The reader can verify it easily.

Lemma 1. *In the prism graph G_0, a Hamilton cycle K contains exactly two edges of M, and these two edges must be adjacent with respect to M. In particular, no Hamilton cycle can contain any two non-adjacent edges of M. If a cycle K contains $|M| - 1$ edges of M, then K has length $n - 1$. In other words, it must miss one vertex.*

Using Lemma 1 we construct a Tutte fragment as follows: taking any two non-adjacent edges of M, say, x_1y_1 and x_iy_i, where $3 \le i \le 2k$, we replace vertex x_1 with a triangle (a complete graph of order 3) $T_1 = u_1v_1w_1$ and x_i with another triangle $T_2 = u_2v_2w_2$, and replace the six edges $x_{2k+1}x_1, x_1y_1, x_1x_2, x_{i-1}x_i, x_iy_i$ and x_ix_{i+1} with edges $x_{2k+1}u_1$, w_1y_1, v_1x_2, $x_{i-1}u_2$, w_2y_i and v_2x_{i+1} respectively. Then one can easily see that no Hamilton cycle of this new graph contains the edges u_1v_1 and u_2v_2 simultaneously. Next, we subdivide the edge u_1v_1 with vertex u and subdivide the edge u_2v_2 with vertex v, and connect u and v with edge uv. Then every Hamilton cycle in this new graph G' must contain edge uv. Indeed G' is hamiltonian, so edge uv is critical and thus the graph $H = G' - v$ is a Tutte fragment. The fragment H has $4k + 7$ vertices. See Fig. 2 for an illustration.

For $k = 2$, there is essentially only one pair of non-adjacent edges of M, and therefore only one non-isomorphic Tutte fragment in this case. This is the one that Tutte used for his construction in [6]. In general, for $k \ge 2$, we have $k - 1$ non-isomorphic Tutte fragments. For example, for $k = 3$, we have two such fragments, by taking $i = 3$ and $i = 4$ respectively.

We summarize the above in the following lemma, which guarantees the existence of infinitely many Tutte fragments and will be used in the proof of our main result. We don't know the exact number of Tutte fragments for a given order n since there may be other ways of constructing different Tutte fragments.

Lemma 2. *For any integer $k \ge 2$, there are at least $k - 1$ non-isomorphic Tutte fragments of order $4k + 7$.*

Let H be the Tutte fragment with $k = 2$, and replace three vertices of K_4 with copies of H, with all pseudo-critical edges incident with the fourth vertex of K_4; we get Tutte's graph, a counterexample to Tait's conjecture. This graph has 46 vertices. See Fig. 1.

Using the fact that the graph $G_0 = C_5 \times K_2$ has no Hamilton cycle containing both edges x_1y_1 and x_3y_3 simultaneously and the above fragment one can construct a smaller 3-connected cubic planar non-hamiltonian graph, see [3]. Let H_1 and H_2 be two distinct copies of H, and replace x_1 with H_1, and x_3 with H_2, respectively, such that the critical vertex of H_1 connects to y_1 and the critical vertex of H_2 connects to y_3. This final graph is non-hamiltonian since otherwise we would have a Hamilton cycle of G_0 containing both edges x_1y_1 and x_3y_3 by contracting H_1 back to x_1 and H_2 back to x_3. See Fig. 3. This non-hamiltonian 3-connected planar cubic graph has 38 vertices. It is proved in [3] that any 3-connected planar cubic graph with fewer than 38 vertices is hamiltonian.

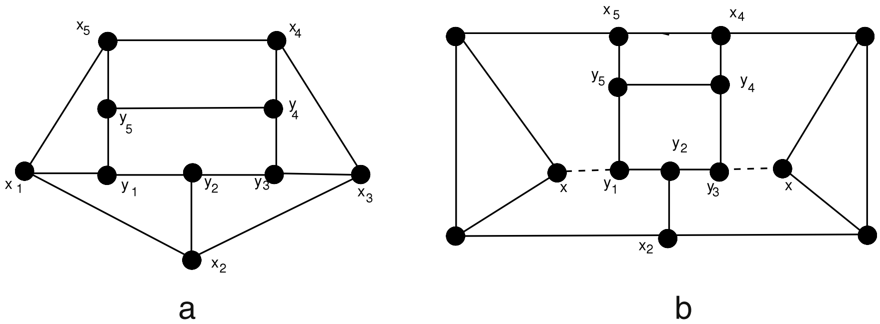

Fig. 3. (a) The pentagonal prism G. (b) The graph $G[\{x_1, x_3\}, H]$, where H is the Tutte fragment in Fig. 1; only boundary vertices of H are shown.

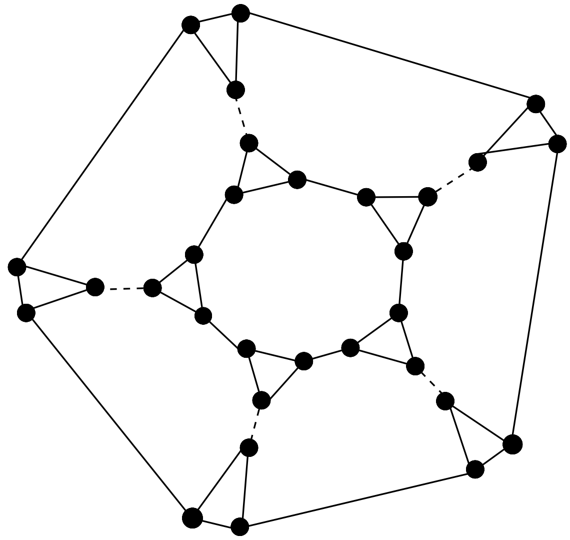

Fig. 4. The graph $G[V(G), H]$, where G is the pentagonal prism and H is the Tutte fragment in Fig. 1; only boundary vertices of H are shown. Pseudo-critical edges are dashed.

3. Main result

We are going to construct 3-connected cubic planar graphs without long cycles. To be precise, we prove the following result.

Theorem. *There exist* 3*-connected cubic planar graphs of order* n *with the longest cycle of length at most* $n - \sqrt{n + \frac{49}{4}} + \frac{5}{2}$ *for infinitely many integers* n*.*

Proof. We construct our graph G as follows. Let H be a Tutte fragment of order $4k + 7$, whose existence is guaranteed by Lemma 2, and let $G_0 = C_{2k'+1} \times K_2$ be the prism graph of order $4k' + 2$. Let M be the perfect matching of G_0 as described in Lemma 1. We construct our graph $G = G_0[V(G_0), H]$ as follows. For each vertex u of G_0 we take a copy $H(u)$ of H, and we replace u with $H(u)$. If xy is an edge in M, we then connect the two critical vertices of $H(x)$ and $H(y)$, and thus create a pseudo-critical edge $\phi(xy)$ corresponding to xy. If uv is an edge not in M, then we connect a non-critical boundary vertex of $H(u)$ with a non-critical boundary vertex of $H(v)$. We might have more than one way of choosing the boundary vertices (since we might have two boundary vertices available sometime), but this does not matter. All choices work for our purpose. See Fig. 4 for an example. For convenience we say that two pseudo-critical edges $\phi(xy)$ and $\phi(x'y')$ are adjacent with respect to M if xy and $x'y'$ are adjacent with respect to M.

Let C be a longest cycle of G. Notice that the $|M|$ pseudo-critical edges separate G into two connected components of equal order and thus we only need to consider the following three cases.

Case 1: The cycle C contains two non-adjacent pseudo-critical edges with respect to M. In this case, C must exclude at least one entire copy of H, say $H(x)$, by Lemma 1 and thus excludes at least $|H| = 4k + 7$ vertices. Indeed, we can do a little bit better. Since C excludes an entire copy $H(x)$, the corresponding pseudo-critical edge $\phi(xy)$ is also excluded. Thus at least one vertex of $H(y)$ must be excluded. This implies that C excludes at least $4k + 8$ vertices.

Case 2: The cycle C only contains two adjacent pseudo-critical edges with respect to M and misses all other $2k'-1$ pseudo-critical edges. In this case, a pseudo-critical edge missed by C connects two copies of H, and C excludes at least one vertex from each copy so it excludes at least $4k'-2$ vertices.

Case 3: The cycle C does not contain any pseudo-critical edge. In this case C excludes at least half of the vertices of G.

From above we see that a longest cycle of G excludes at least $\min\{4k+8, 4k'-2\}$ vertices. Letting $k'=k+3$, we have $n=(4k'+2)(4k+7)=(4k+14)(4k+7)$ and $\min\{4k+8, 4k'-2\}=4k+8=\sqrt{n+\frac{49}{4}}-\frac{5}{2}$. This proves our theorem. □

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